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Representations and Clebsch–Gordan coefficients for the Jordanian quantum algebra $\mathcal{U}_h(sl(2))$

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Abstract. Representation theory for the Jordanian quantum algebra $\mathcal{U}_h(sl(2))$ is developed. Closed form expressions are given for the action of the generators of $\mathcal{U}_h(sl(2))$ on the basis vectors of finite-dimensional irreducible representations. It is shown how representation theory of $\mathcal{U}_h(sl(2))$ has a close connection to the combinatorial identities involving summation formulae. A general formula is obtained for the Clebsch–Gordan coefficients $\mathcal{C}^{j_1,j_2,j}_{n_1,n_2,m}(h)$ of $\mathcal{U}_h(sl(2))$. These Clebsch–Gordan coefficients are shown to coincide with those of $su(2)$ for $n_1+n_2 \leqslant m$, but for $n_1+n_2 > m$ they are, in general, a non-zero monomial in $h^{n_1+n_2-m}$.

1. Introduction

Viewing a quantum group as a quantum automorphism group acting on a non-commuting space [1–3], one often requires the extra condition of the existence of a central determinant in the quantum matrix group. For two-by-two matrices, this condition restricts the quantum groups essentially to only two classes [4], namely the standard $SL_q(2)$ quantum group and the Jordanian $SL_h(2)$ quantum group. The quantum group $SL_h(2)$ was introduced in [5], and the corresponding quantum algebra (or quantized universal enveloping algebra) $\mathcal{U}_h(sl(2))$ was given in [6]. A universal R-matrix for $\mathcal{U}_h(sl(2))$ was constructed in [7].

The main object of this paper is to develop representation theory of the Jordanian quantum algebra $\mathcal{U}_h(sl(2))$, and in particular construct Clebsch–Gordan coefficients. The finite-dimensional highest weight representations of $\mathcal{U}_h(sl(2))$ were given in [8], first by a direct construction, and then by factorizing the corresponding Verma module. In [8] the action of the $\mathcal{U}_h(sl(2))$ generators on a finite-dimensional representation was not given explicitly. An important construction was developed by Abdesselam *et al* [9]: they gave a nonlinear relation between the generators of $\mathcal{U}_h(sl(2))$ and the classical generators of $sl(2)$. As a consequence of this relation, they obtained expressions for the action of the $\mathcal{U}_h(sl(2))$ generators H, X and Y (see the following section for their definition) on basis vectors of the finite-dimensional irreducible representations. These expressions are in closed form, except for the action of the generator Y. Using this nonlinear map, Aizawa [10] constructed finite- and infinite-dimensional representations of $\mathcal{U}_h(sl(2))$, and considered the tensor product of two representations. Moreover, he gives some examples of Clebsch–Gordan coefficients.

Our present work is motivated by the fact that $sl(2)$ or $su(2)$ representations appear in many physical theories, and often their Clebsch–Gordan coefficients are fundamental in these theories. Since representation theory of $\mathcal{U}_h(sl(2))$ is so closely related to that of $su(2)$, and

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could be used as the algebraic structure underlying deformations of such physical models, an important aspect to study are the Clebsch–Gordan coefficients of $\mathcal{U}_h(sl(2))$. In the present paper it is shown how an explicit formula for $\mathcal{U}_h(sl(2))$ Clebsch–Gordan coefficients can be obtained.

In section 2, we give the defining relations for $\mathcal{U}_h(sl(2))$, and the nonlinear relation between the $sl(2)$ generators and $\mathcal{U}_h(sl(2))$ generators. In section 3, closed forms for the action of the three generators H, X and Y of $\mathcal{U}_h(sl(2))$ acting on the basis vectors of finite-dimensional irreducible representations are determined. For H and X, these expressions correspond to those of [9]; the determination of the explicit action of Y is new and is found using a number of combinatorial identities (lemmas 1 and 2). In section 4 the tensor product of two representations is considered. In this tensor product we show the existence of an auxiliary basis which behaves like the uncoupled basis vectors in the tensor product of two $su(2)$ representations. Using this auxiliary basis, the $\mathcal{U}_h(sl(2))$ Clebsch–Gordan coefficients are easily determined in section 5, and some examples and properties are discussed in section 6.

A curious aspect of the results in this paper (and more generally of $\mathcal{U}_h(sl(2))$ representation theory) is that they are closely related to non-trivial combinatorial identities (see lemmas 1, 2 and 4). The identities needed here have on one side a (definite or indefinite) sum of hypergeometric terms, and a closed form expression on the other. To find closed form expressions for such summations is a problem that can be solved completely algorithmically [11]: for indefinite summations this can be done by means of Gosper's algorithm; for definite summations this is done by means of Zeilberger's algorithm. For both algorithms, programs are available in Maple or Mathematica. In the appendix, we comment on the proofs of these combinatorial identities.

2. Definition and relation to $sl(2)$

The Jordanian quantum algebra $\mathcal{U}_h(sl(2))$ is an associative algebra with unity and generated by X, Y and H, subject to the relations

$$[H,X] = 2\frac{\sinh hX}{h} \qquad [H,Y] = -Y(\cosh hX) - (\cosh hX)Y$$
$$[X,Y] = H. \tag{2.1}$$

Herein, h is the deformation parameter. We shall assume that $|h| < 1$. In the limit $h \to 0$, $\mathcal{U}_h(sl(2))$ reduces to the universal enveloping algebra of $sl(2)$. The Hopf algebra structure of $\mathcal{U}_h(sl(2))$ is given in [6]; here, we are only interested in the comultiplication, which reads

$$\Delta(X) = X \otimes 1 + 1 \otimes X$$
$$\Delta(Y) = Y \otimes \mathrm{e}^{hX} + \mathrm{e}^{-hX} \otimes Y$$
$$\Delta(H) = H \otimes \mathrm{e}^{hX} + \mathrm{e}^{-hX} \otimes H. \tag{2.2}$$

The irreducible finite-dimensional highest weight representations of $\mathcal{U}_h(sl(2))$ can be obtained by using the invertible map from $sl(2)$ to $\mathcal{U}_h(sl(2))$, given in [9]. With the following definitions

$$Z_+ = \frac{2}{h}\tanh\frac{hX}{2} \qquad Z_- = \left(\cosh\frac{hX}{2}\right) Y \left(\cosh\frac{hX}{2}\right) \tag{2.3}$$

it follows that the elements $\{H, Z_+, Z_-\}$ satisfy the commutation relations of a classical $sl(2)$ basis:

$$[H, Z_\pm] = \pm 2 Z_\pm \qquad [Z_+, Z_-] = H. \tag{2.4}$$

The relations (2.3) can be inverted, see the following section, and thus with every finite-dimensional irreducible $sl(2)$ representation there corresponds a finite-dimensional irreducible representation of $\mathcal{U}_h(sl(2))$. These $sl(2)$ representations are labelled by a number j, with $2j$ a non-negative integer; the representation space is $V^{(j)}$ with basis e^j_m, where $m = -j, -j+1, \ldots, j$. The action of $sl(2)$ on this basis is given by

$$He^j_m = 2me^j_m$$
$$Z_\pm e^j_m = \sqrt{(j \mp m)(j \pm m + 1)}\,e^j_{m\pm 1}. \qquad (2.5)$$

For most of the computations in this paper, it is easier to work with another basis of $V^{(j)}$ related to the above basis by

$$v^j_m = \alpha_{j,m} e^j_m \qquad \text{with } \alpha_{j,m} = \sqrt{(j+m)!/(j-m)!}. \qquad (2.6)$$

The matrix elements of the $sl(2)$ generators are then given by

$$Hv^j_m = 2mv^j_m \qquad Z_+ v^j_m = v^j_{m+1} \qquad Z_- v^j_m = (j+m)(j-m+1)v^j_{m-1} \qquad (2.7)$$

where $v^j_{j+1} = 0$. Clearly, if for an operator the matrix elements in the v-basis have been determined, the matrix elements in the e-basis follow immediately using (2.6).

3. Representations of $\mathcal{U}_h(sl(2))$

In this section we wish to give explicit expressions for the matrix elements of H, X and Y in the v-basis. For H, this is trivial, see (2.7). For X, one first determines the action of e^{hX}. From relation (2.3) one finds that

$$e^{hX} = \left(1 + \frac{h}{2} Z_+\right)\left(1 - \frac{h}{2} Z_+\right)^{-1}. \qquad (3.1)$$

Then the action of Z_+ in the v-basis implies

$$e^{hX} v^j_m = v^j_m + 2\sum_{k=1}^{j-m} \left(\frac{h}{2}\right)^k v^j_{m+k}. \qquad (3.2)$$

Thus in this representation one can write $e^{hX} = 1 + N_1$, with N_1 a nilpotent matrix. Then $hX = \log(1 + N_1) = N_1 - N_1^2/2 + N_1^3/3 - \cdots$, and one obtains the following action of X in the representation space $V^{(j)}$:

$$Xv^j_m = \sum_{k=0}^{\lfloor (j-m-1)/2 \rfloor} \frac{(h/2)^{2k}}{2k+1} v^j_{m+1+2k}. \qquad (3.3)$$

Up to a scaling of the basis vectors, (3.3) coincides with [9, equation (23)]. The action of Y is more difficult to determine explicitly; in [9, equation (35)] an expression is given but the matrix elements still involve a single sum. Here we shall show that it can actually be given in closed form. Let us use the relation

$$Y = \left(\cosh \frac{hX}{2}\right)^{-1} Z_- \left(\cosh \frac{hX}{2}\right)^{-1} \qquad (3.4)$$

and first determine the matrix form of $(\cosh(hX/2))^{-1}$ in the v-basis. In this basis of $V^{(j)}$, one can write

$$\left(\cosh \frac{hX}{2}\right)^2 = \frac{1}{2}(1 + \cosh hX) = \frac{1}{2}\left(1 + \frac{e^{hX} + e^{-hX}}{2}\right) = 1 + N_2 \qquad (3.5)$$

where, by (3.2), N_2 is again a nilpotent matrix whose matrix elements follow from those of e^{hX}:

$$N_2 v_m^j = \sum_{k=1}^{\lfloor (j-m)/2 \rfloor} (h/2)^{2k} v_{m+2k}^j. \tag{3.6}$$

Then, in this representation

$$\left(\cosh \frac{hX}{2}\right)^{-1} = (1+N_2)^{-1/2} = \sum_{n=0}^{\infty} (-1)^n (1/2)_n \frac{N_2^n}{n!} \tag{3.7}$$

where $(a)_n$ is the notation for the Pochhammer symbol:

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1) & \text{if } n = 1, 2, \ldots \\ 1 & \text{if } n = 0. \end{cases} \tag{3.8}$$

Using the explicit powers of N_2 in the action of (3.7) on v_m^j, the contributions to the coefficient of v_{m+2k}^j $(k > 0)$ in $(1+N_2)^{-1/2} v_m^j$ are

$$\left(\frac{h}{2}\right)^{2k} \sum_{n=1}^{k} (-1)^n \frac{(1/2)_n}{n!} \binom{k-1}{n-1}. \tag{3.9}$$

Next, we use

Lemma 1. For $k > 0$ integer,

$$\sum_{n=1}^{k} (-1)^n \frac{(1/2)_n}{n!} \binom{k-1}{n-1} = -\frac{1}{2^{2k-1}} \frac{(2k-2)!}{k!(k-1)!}.$$

As a consequence, one finds the explicit action of $(\cosh(hX/2))^{-1}$ in the v-basis:

$$\left(\cosh \frac{hX}{2}\right)^{-1} v_m^j = v_m^j - 2 \sum_{k=1}^{\lfloor (j-m)/2 \rfloor} t_k \left(\frac{h}{4}\right)^{2k} v_{m+2k}^j \tag{3.10}$$

with $t_k = (2k-2)!/k!(k-1)!$. Using (3.4), (3.10) and (2.7), one determines the action of Y,

$$\begin{aligned} Y v_m^j &= \left(\cosh \frac{hX}{2}\right)^{-1} Z_- \left(v_m^j - 2 \sum_{k\geqslant 1} t_k (h/4)^{2k} v_{m+2k}^j \right) \\ &= \left(\cosh \frac{hX}{2}\right)^{-1} \Bigg((j+m)(j-m+1) v_{m-1}^j \\ &\quad - 2 \sum_{k\geqslant 1} t_k (h/4)^{2k} (j+m+2k)(j-m-2k+1) v_{m+2k-1}^j \Bigg) \\ &= (j+m)(j-m+1) \left(v_{m-1}^j - 2 \sum_{l\geqslant 1} t_l (h/4)^{2l} v_{m+2l-1}^j \right) \\ &\quad - 2 \sum_{k\geqslant 1} t_k (h/4)^{2k} (j+m+2k)(j-m-2k+1) \\ &\quad \times \left(v_{m+2k-1}^j - 2 \sum_{l\geqslant 1} t_l (h/4)^{2l} v_{m+2k+2l-1}^j \right). \end{aligned} \tag{3.11}$$

In this last expression, the coefficient of v^j_{m-1} is $(j+m)(j-m+1)$. The coefficient of v^j_{m+1} is

$$-2(j+m)(j-m+1)t_1h^2/16 - 2(j+m+2)(j-m-1)t_1h^2/16 \\ = h^2/4 - (j-m)(j+m+1)h^2/4. \tag{3.12}$$

Finally, we determine the coefficient of v^j_{m+2s-1} ($s \geqslant 2$) in (3.11); it reads

$$-2(j+m)(j-m+1)t_s(h/4)^{2s} - 2(j+m+2s)(j-m-2s+1)t_s(h/4)^{2s} \\ +4(h/4)^{2s}\sum_{k=1}^{s-1}(j+m+2k)(j-m-2k+1)t_kt_{s-k}. \tag{3.13}$$

In order to give a closed form for this coefficient, we need to find an expression for $\sum_k t_kt_{s-k}k^n$ with $n = 0$, 1 and 2. This is done with the help of the following lemma.

Lemma 2. Let $s \geqslant 2$ be integer, then

$$\sum_{k=1}^{s-1}\frac{(2k-2)!}{k!(k-1)!}\frac{(2s-2k-2)!}{(s-k)!(s-k-1)!}k^n = \begin{cases} \dfrac{(2s-2)!}{s!(s-1)!} & \text{if } n=0 \\ \dfrac{(2s-2)!}{2(s-1)!^2} & \text{if } n=1 \\ \dfrac{s(2s-2)!}{2(s-1)!^2} - 4^{s-2} & \text{if } n=2. \end{cases}$$

Using these three results, it follows from (3.13) that the coefficient of v^j_{m+2s-1} ($s \geqslant 2$) is $4^s(h/4)^{2s} = (h/2)^{2s}$. Thus we finally obtain the explicit action of Y on the v-basis:

$$Yv^j_m = (j+m)(j-m+1)v^j_{m-1} - (j-m)(j+m+1)\left(\frac{h}{2}\right)^2 v^j_{m+1} \\ + \sum_{s=1}^{\lfloor (j-m+1)/2\rfloor}\left(\frac{h}{2}\right)^{2s} v^j_{m-1+2s}. \tag{3.14}$$

Thus the first equation of (2.7) together with equations (3.3) and (3.14) determine the action of the generators H, X and Y of $\mathcal{U}_h(sl(2))$ on the finite-dimensional irreducible representations in closed form.

4. Tensor product of two representations

The action of any of the generators H, X or Y on the tensor product of two representations is determined by the comultiplication. The comultiplication rule on H, X and Y induces a comultiplication on H, $Z_\pm$. The purpose of this section is to show that the tensor product $V^{(j_1)} \otimes V^{(j_2)}$ decomposes into a direct sum of representations isomorphic to $V^{(j)}$, where $j = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|$, and to determine the Clebsch–Gordan coefficients in

$$e^{(j_1j_2)j}_m = \sum_{n_1,n_2}\mathcal{C}^{j_1,j_2,j}_{n_1,n_2,m}(h)e^{j_1}_{n_1}\otimes e^{j_2}_{n_2} \tag{4.1}$$

such that $e^{(j_1j_2)j}_m$ is a standard e-basis of $V^{(j)}$, i.e.

$$\Delta(H)e^{(j_1j_2)j}_m = 2me^{(j_1j_2)j}_m \\ \Delta(Z_\pm)e^{(j_1j_2)j}_m = \sqrt{(j\mp m)(j\pm m+1)}e^{(j_1j_2)j}_{m\pm1}. \tag{4.2}$$

The first step towards this goal is to find expressions of $\Delta(H)$ and $\Delta(Z_+)$ in terms of H and Z_+. For H, the problem is easy, and follows from (2.2) and (3.1), see also [10, equation (3.2)]:

$$\Delta(H) = H \otimes e^{hX} + e^{-hX} \otimes H = H \otimes 1 + 1 \otimes H + 2H \otimes \sum_{n=1}^{\infty} \left(\frac{hZ_+}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{-hZ_+}{2}\right)^n \otimes 2H. \tag{4.3}$$

To determine $\Delta(Z_+)$, denote $t = e^{hX}$ and $z = (h/2)Z_+$. Then $z = (t-1)(t+1)^{-1}$ and $t = (1+z)(1-z)^{-1}$. Moreover,

$$\Delta(t) = t \otimes t = \left(1 + 2\sum_{k=1}^{\infty} z^k\right) \otimes \left(1 + 2\sum_{k=1}^{\infty} z^k\right). \tag{4.4}$$

Thus $\Delta(z) = \sum_{k,l=0}^{\infty} \lambda_{k,l} z^k \otimes z^l$, and the coefficients can be obtained from

$$(1 \otimes 1 - \Delta(z))\Delta(t) = 1 \otimes 1 + \Delta(z)$$

which must hold since $(1-z)t = (1+z)$. This leads to

$$\Delta(z) = (1 \otimes z + z \otimes 1)(1 \otimes 1 - z \otimes z + z^2 \otimes z^2 - \cdots)$$

and we obtain

$$\Delta(Z_+) = (1 \otimes Z_+ + Z_+ \otimes 1)\left(\sum_{n=0}^{\infty} (-h^2/4)^n Z_+^n \otimes Z_+^n\right). \tag{4.5}$$

An explicit expression for $\Delta(Z_-)$ is much more complicated (see, e.g., equation (5.3) of [10]), but we do not need it here.

In the tensor product space $V^{(j_1)} \otimes V^{(j_2)}$, we now define an auxiliary basis that is expressed in terms of the v-basis $v_{m_1}^{j_1} \otimes v_{m_2}^{j_2}$. Define the coefficients (see (3.8) for the notation of Pochhammer symbols)

$$b_{k,l}^{m_1,m_2} = \begin{cases} \dfrac{(-2m_1-k)_l(-2m_2-l)_k}{k!l!} & \text{if } k \geqslant 0 \text{ and } l \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

and

$$a_{k,l}^{m_1,m_2} = (-1)^k (h/2)^{k+l} (b_{k,l}^{m_1,m_2} - b_{k-1,l-1}^{m_1,m_2}). \tag{4.7}$$

The auxiliary vectors are defined as follows:

$$w_{m_1,m_2}^{j_1,j_2} = \sum_{k=0}^{j_1-m_1} \sum_{l=0}^{j_2-m_2} a_{k,l}^{m_1,m_2} v_{m_1+k}^{j_1} \otimes v_{m_2+l}^{j_2}. \tag{4.8}$$

Clearly, they also form a basis for $V^{(j_1)} \otimes V^{(j_2)}$ since the relation between the vectors $w_{m_1,m_2}^{j_1,j_2}$ and the vectors $v_{m_1}^{j_1} \otimes v_{m_2}^{j_2}$ is given by an upper triangular matrix with 1's on the diagonal. This auxiliary basis has been introduced because the action of $\Delta(H)$ and $\Delta(Z_\pm)$ on it is simple. The idea of introducing such an auxiliary basis comes from [10]; however, the coefficients used in [10, equation (3.9)] are single sum expressions and no closed forms.

Proposition 3. The action of $\Delta(H)$ on the auxiliary basis vectors is given by

$$\Delta(H)w^{j_1,j_2}_{m_1,m_2} = 2(m_1+m_2)w^{j_1,j_2}_{m_1,m_2}.$$

Proof. To prove this, use (4.3), (4.8), and the explicit action of H and Z_+ on the v-basis:

$$\begin{aligned}\Delta(H)w^{j_1,j_2}_{m_1,m_2} &= \Delta(H)\Bigg(\sum_{k,l\geqslant 0} a^{m_1,m_2}_{k,l} v^{j_1}_{m_1+k}\otimes v^{j_2}_{m_2+l}\Bigg)\\ &= \sum_{k,l\geqslant 0} a^{m_1,m_2}_{k,l}\Bigg(2(m_1+k+m_2+l)v^{j_1}_{m_1+k}\otimes v^{j_2}_{m_2+l} + 4(m_1+k)v^{j_1}_{m_1+k}\\ &\quad\otimes \sum_{n\geqslant 1}(h/2)^n v^{j_2}_{m_2+l+n} + 4(m_2+l)\sum_{n\geqslant 1}(-h/2)^n v^{j_1}_{m_1+k+n}\otimes v^{j_2}_{m_2+l}\Bigg)\\ &= 2(m_1+m_2)w^{j_1,j_2}_{m_1,m_2} + 2\sum_{k,l\geqslant 0} v^{j_1}_{m_1+k}\otimes v^{j_2}_{m_2+l}\Bigg((k+l)a^{m_1,m_2}_{k,l} + 2(m_1+k)\\ &\quad\times\sum_{n=1}^{l}(h/2)^n a^{m_1,m_2}_{k,l-n} + 2(m_2+l)\sum_{n=1}^{l}(-h/2)^n a^{m_1,m_2}_{k-n,l}\Bigg). \end{aligned} \tag{4.9}$$

Thus the proposition is proved provided we can show that the coefficient of $v^{j_1}_{m_1+k}\otimes v^{j_2}_{m_2+l}$ in the last summation is zero. Consider first

$$\sum_{n=1}^{l}(h/2)^n a^{m_1,m_2}_{k,l-n} = (-1)^k(h/2)^{k+l}\sum_{n=1}^{l}(b^{m_1,m_2}_{k,l-n} - b^{m_1,m_2}_{k-1,l-n-1}). \tag{4.10}$$

We shall show that this last sum can be performed and yields

$$\sum_{n=1}^{l}(b^{m_1,m_2}_{k,l-n} - b^{m_1,m_2}_{k-1,l-n-1}) = \frac{2m_1+k-l+1}{2m_1+k} b^{m_1,m_2}_{k,l-1}. \tag{4.11}$$

Using the symmetry $b^{m_1,m_2}_{k,l} = b^{m_2,m_1}_{l,k}$, one can use the same result (4.11) to find

$$\sum_{n=1}^{l}(-h/2)^n a^{m_1,m_2}_{k-n,l} = (-1)^k(h/2)^{k+l}\left(\frac{2m_2+l-k+1}{2m_2+l}\right) b^{m_1,m_2}_{k-1,l}. \tag{4.12}$$

Then the coefficient of $v^{j_1}_{m_1+k}\otimes v^{j_2}_{m_2+l}$ in the summation part of (4.9) is, up to a factor $(-1)^k(h/2)^{k+l}$, equal to

$$\begin{aligned}&(k+l)(b^{m_1,m_2}_{k,l} - b^{m_1,m_2}_{k-1,l-1}) + 2(m_1+k)\left(\frac{2m_1+k-l+1}{2m_1+k}\right) b^{m_1,m_2}_{k,l-1}\\ &\quad +2(m_2+l)\left(\frac{2m_2+l-k+1}{2m_2+l}\right) b^{m_1,m_2}_{k-1,l}\end{aligned} \tag{4.13}$$

and using the definition of the coefficients $b^{m_1,m_2}_{k,l}$ this is trivially shown to be zero. What remains to be proved is (4.11). This follows from the following lemma.

Lemma 4. Let α, β, γ be arbitrary parameters, $l\geqslant 0$ integer, and

$$g(l) = \frac{(\alpha)_l(\beta)_l}{(l+1)!(\gamma)_l}((\gamma-\alpha-\beta)l+\gamma-1-\alpha\beta).$$

Then for $n\geqslant 0$ integer,

$$\sum_{l=0}^{n} g(l) = \gamma - 1 - \frac{(\alpha)_{n+1}(\beta)_{n+1}}{(n+1)!(\gamma)_n}.$$

The sum (4.11) then follows from this lemma by putting $\alpha = 1-k-2m_1$, $\beta = 1+2m_2$ and $\gamma = 2-k+2m_2$. This completes the proof of proposition 3. □

Proposition 5. The action of $\Delta(Z_+)$ on the auxiliary basis vectors is given by

$$\Delta(Z_+)w^{j_1,j_2}_{m_1,m_2} = w^{j_1,j_2}_{m_1+1,m_2} + w^{j_1,j_2}_{m_1,m_2+1} \tag{4.14}$$

where $w^{j_1,j_2}_{m_1,m_2}$ is interpreted as zero if one of the indices $m_i > j_i$.

Proof. This is proved by direct computation, and does not involve any combinatorial identities. The left-hand side of (4.14) yields

$$\begin{aligned}(1\otimes Z_+ + Z_+\otimes 1)&\left(\sum_{k,l\geqslant 0}\sum_{n\geqslant 0} a^{m_1,m_2}_{k,l}(-h^2/4)^n v^{j_1}_{m_1+k+n}\otimes v^{j_2}_{m_2+l+n}\right)\\ &= (1\otimes Z_+ + Z_+\otimes 1)\left\{\sum_{k,l\geqslant 0}\left(\sum_{n\geqslant 0} a^{m_1,m_2}_{k-n,l-n}(-h^2/4)^n\right)v^{j_1}_{m_1+k}\otimes v^{j_2}_{m_2+l}\right\}.\end{aligned} \tag{4.15}$$

From definition (4.7) of the coefficients $a^{m_1,m_2}_{k,l}$ it follows that

$$\begin{aligned}\sum_{n\geqslant 0} a^{m_1,m_2}_{k-n,l-n}(-h^2/4)^n &= (-1)^k(h/2)^{k+l}\sum_{n\geqslant 0}(b^{m_1,m_2}_{k-n,l-n} - b^{m_1,m_2}_{k-n-1,l-n-1})\\ &= (-1)^k(h/2)^{k+l} b^{m_1,m_2}_{k,l}.\end{aligned} \tag{4.16}$$

Putting this back in (4.15) gives

$$\begin{aligned}\sum_{k,l\geqslant 0}(-1)^k(h/2)^{k+l}b^{m_1,m_2}_{k,l}v^{j_1}_{m_1+k}\otimes v^{j_2}_{m_2+l+1} + \sum_{k,l\geqslant 0}(-1)^k(h/2)^{k+l}b^{m_1,m_2}_{k,l}v^{j_1}_{m_1+k+1}\otimes v^{j_2}_{m_2+l}\\ = \sum_{k,l\geqslant 0}(-1)^k(h/2)^{k+l-1}(b^{m_1,m_2}_{k,l-1} - b^{m_1,m_2}_{k-1,l})v^{j_1}_{m_1+k}\otimes v^{j_2}_{m_2+l}.\end{aligned} \tag{4.17}$$

On the other hand, the right-hand side of (4.14) leads to

$$\begin{aligned}\sum_{k,l\geqslant 0}(a^{m_1,m_2+1}_{k,l-1} - a^{m_1+1,m_2}_{k-1,l})v^{j_1}_{m_1+k}\otimes v^{j_2}_{m_2+l} &= \sum_{k,l\geqslant 0}(-1)^k(h/2)^{k+l-1}\\ \times(b^{m_1,m_2+1}_{k,l-1} - b^{m_1,m_2+1}_{k-1,l-2} - b^{m_1+1,m_2}_{k-1,l} + b^{m_1+1,m_2}_{k-2,l-1})&v^{j_1}_{m_1+k}\otimes v^{j_2}_{m_2+l}.\end{aligned} \tag{4.18}$$

So, it remains to show that

$$b^{m_1,m_2}_{k,l-1} - b^{m_1,m_2}_{k-1,l} = b^{m_1,m_2+1}_{k,l-1} - b^{m_1,m_2+1}_{k-1,l-2} - b^{m_1+1,m_2}_{k-1,l} + b^{m_1+1,m_2}_{k-2,l-1} \tag{4.19}$$

and this follows trivially from the definition (4.6) of the b-coefficients. □

Proposition 5. The action of $\Delta(Z_-)$ on the auxiliary basis vectors is given by

$$\Delta(Z_-)w^{j_1,j_2}_{m_1,m_2} = (j_1+m_1)(j_1-m_1+1)w^{j_1,j_2}_{m_1-1,m_2} + (j_2+m_2)(j_2-m_2+1)w^{j_1,j_2}_{m_1,m_2-1}. \tag{4.20}$$

Proof. The proof of this property is rather long and technical. Let h, $z_\pm$ be the standard basis of $sl(2)$, with

$$[h, z_\pm] = \pm 2z_\pm \qquad [z_+, z_-] = h. \tag{4.21}$$

$sl(2)$ has an action on the basis elements v^j_m, given by the same expressions as in (2.7). The standard comultiplication for $sl(2)$, $\delta(x) = x\otimes 1 + 1\otimes x$ for every $x\in sl(2)$, induces an action on elements of $V^{(j_1)}\otimes V^{(j_2)}$:

$$\begin{aligned}\delta(h)v^{j_1}_{m_1}\otimes v^{j_2}_{m_2} &= 2(m_1+m_2)v^{j_1}_{m_1}\otimes v^{j_2}_{m_2}\\ \delta(z_+)v^{j_1}_{m_1}\otimes v^{j_2}_{m_2} &= v^{j_1}_{m_1+1}\otimes v^{j_2}_{m_2} + v^{j_1}_{m_1}\otimes v^{j_2}_{m_2+1}\\ \delta(z_-)v^{j_1}_{m_1}\otimes v^{j_2}_{m_2} &= (j_1+m_1)(j_1-m_1+1)v^{j_1}_{m_1+1}\\ &\quad\otimes v^{j_2}_{m_2} + (j_2+m_2)(j_2-m_2+1)v^{j_1}_{m_1}\otimes v^{j_2}_{m_2+1}.\end{aligned} \tag{4.22}$$

Thus the action of $\delta(h)$ and $\delta(z_+)$ on the basis $v^{j_1}_{m_1} \otimes v^{j_2}_{m_2}$ is the same as the action of $\Delta(H)$ and $\Delta(Z_+)$ on the auxiliary basis $w^{j_1,j_2}_{m_1,m_2}$, by propositions 3 and 5. We shall show that this holds for Z_- too. First of all, for $sl(2)$ we know how the tensor product decomposes, so let

$$v^{(j_1j_2)j}_m = \sum_{m_1+m_2=m} c^{j_1,j_2,j}_{m_1,m_2,m} v^{j_1}_{m_1} \otimes v^{j_2}_{m_2} \tag{4.23}$$

where $j \in J = \{j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|\}$, and $c^{j_1,j_2,j}_{m_1,m_2,m}$ are the Clebsch–Gordan coefficients in the v-basis, related to the usual $su(2)$ Clebsch–Gordan coefficients by

$$c^{j_1,j_2,j}_{m_1,m_2,m} = \frac{\alpha_{j,m}}{\alpha_{j_1,m_1}\alpha_{j_2,m_2}} C^{j_1,j_2,j}_{m_1,m_2,m} \tag{4.24}$$

see (2.6). Then there holds

$$\begin{aligned} \delta(h)v^{(j_1j_2)j}_m &= 2m v^{(j_1j_2)j}_m \\ \delta(z_+)v^{(j_1j_2)j}_m &= v^{(j_1j_2)j}_{m+1} \\ \delta(z_-)v^{(j_1j_2)j}_m &= (j+m)(j-m+1)v^{(j_1j_2)j}_{m-1}. \end{aligned} \tag{4.25}$$

We also define

$$w^{(j_1j_2)j}_m = \sum_{m_1+m_2=m} c^{j_1,j_2,j}_{m_1,m_2,m} w^{j_1,j_2}_{m_1,m_2} \tag{4.26}$$

and by the remark that the action of $\delta(h)$ and $\delta(z_+)$ on the basis $v^{j_1}_{m_1} \otimes v^{j_2}_{m_2}$ is the same as the action of $\Delta(H)$ and $\Delta(Z_+)$ on the auxiliary basis $w^{j_1,j_2}_{m_1,m_2}$, it follows that

$$\delta(H)w^{(j_1j_2)j}_m = 2m w^{(j_1j_2)j}_m \qquad \delta(Z_+)w^{(j_1j_2)j}_m = w^{(j_1j_2)j}_{m+1}. \tag{4.27}$$

It remains to find the action of $\Delta(Z_-)$ on $w^{(j_1j_2)j}_m$; let us denote it by

$$\Delta(Z_-)w^{(j_1j_2)j}_m = \sum_{j'}\sum_{m'} \mu^{j,j'}_{m,m'} w^{(j_1j_2)j'}_{m'} \tag{4.28}$$

with $\mu^{j,j'}_{m,m'}$ the coefficients to be determined. By acting with the relation

$$\Delta(H)\Delta(Z_-) - \Delta(Z_-)\Delta(H) = -2\Delta(Z_-) \tag{4.29}$$

on $w^{(j_1j_2)j}_m$, it follows that the coefficients $\mu^{j,j'}_{m,m'}$ are zero unless $m' = m - 1$. Write $\nu^{j,j'}_m$ for $\mu^{j,j'}_{m,m-1}$; then we have so far

$$\Delta(Z_-)w^{(j_1j_2)j}_m = \sum_{j'} \nu^{j,j'}_m w^{(j_1j_2)j'}_{m-1} \tag{4.30}$$

where $j' \in J$ such that $j' \geqslant |m-1|$. Next, we use the relation

$$\Delta(Z_+)\Delta(Z_-) - \Delta(Z_-)\Delta(Z_+) = \Delta(H). \tag{4.31}$$

Acting with (4.31) on $w^{(j_1j_2)j}_m$ yields

$$\text{for } j \neq j': \qquad \nu^{j,j'}_m = \nu^{j,j'}_{m+1} \tag{4.32}$$

$$\text{for } j' = j: \qquad \nu^{j,j}_m - \nu^{j,j}_{m+1} = 2m. \tag{4.33}$$

In particular, by acting with (4.31) on $w^{(j_1j_2)j}_j$, one finds that $\nu^{j,j}_j = 2j$, and that $\nu^{j,j'}_j = 0$ for $j' > j$. Now (4.33) is a difference equation in m with boundary condition $\nu^{j,j}_j = 2j$, so it has a unique solution given by

$$\nu^{j,j}_m = (j+m)(j-m+1). \tag{4.34}$$

From (4.32) and $\nu_j^{j,j'} = 0$ for $j' > j$ it follows that $\nu_m^{j,j'} = 0$ for all $j' > j$. Using this and acting with (4.31) on $w_{j-1}^{(j_1j_2)j}$ implies that $\nu_{j-1}^{j,j-1} = 0$, and thus by (4.32) that $\nu_m^{j,j-1} = 0$ for all m. Similarly, acting with (4.31) on $w_{j-2}^{(j_1j_2)j}$ implies that $\nu_{j-2}^{j,j-2} = 0$, and thus by (4.32) that $\nu_m^{j,j-2} = 0$ for all m. One can continue and thus show by induction that $\nu_m^{j,j'} = 0$ also for all $j' < j$. The final result is that

$$\Delta(Z_-)w_m^{(j_1,j_2)j} = (j+m)(j-m+1)w_{m-1}^{(j_1,j_2)j}. \tag{4.35}$$

In other words, $\Delta(Z_-)$ has on the basis $w_m^{(j_1,j_2)j}$ the same action as $\delta(z_-)$ on the basis $v_m^{(j_1j_2)j}$. By relations (4.23) and (4.26), it follows that the action of $\Delta(Z_-)$ on the basis $w_{m_1,m_2}^{j_1,j_2}$ is the same as the action of $\delta(z_-)$ on the basis $v_{m_1}^{j_1} \otimes v_{m_2}^{j_2}$. This proves proposition 6. □

5. Clebsch–Gordan coefficients for $\mathcal{U}_h(sl(2))$

Let us first normalize the auxiliary basis $w_{m_1,m_2}^{j_1,j_2}$ as follows:

$$e_{m_1,m_2}^{j_1,j_2} = w_{m_1,m_2}^{j_1,j_2}/(\alpha_{j_1,m_1}\alpha_{j_2,m_2}). \tag{5.1}$$

Then (4.8) and (2.6) imply that

$$e_{m_1,m_2}^{j_1,j_2} = \sum_{k,l\geqslant 0} A_{k,l}^{m_1,m_2} e_{m_1+k}^{j_1} \otimes e_{m_2+l}^{j_2} \tag{5.2}$$

where

$$A_{k,l}^{m_1,m_2} = a_{k,l}^{m_1,m_2}\frac{\alpha_{j_1,m_1+k}\alpha_{j_2,m_2+l}}{\alpha_{j_1,m_1}\alpha_{j_2,m_2}}. \tag{5.3}$$

Note that the A-coefficients depend implicitly also on j_1 and j_2: $A_{k,l}^{m_1,m_2} = 0$ unless m_1 and m_1+k belong to $\{-j_1, -j_1+1, \ldots, j_1\}$ and m_2 and m_2+l belong to $\{-j_2, -j_2+1, \ldots, j_2\}$. From the previous section it follows that the following relations hold:

$$\begin{aligned}
&\Delta(H)e_{m_1,m_2}^{j_1,j_2} = 2(m_1+m_2)e_{m_1,m_2}^{j_1,j_2}\\
&\Delta(Z_+)e_{m_1,m_2}^{j_1,j_2} = \sqrt{(j_1-m_1)(j_1+m_1+1)}e_{m_1+1,m_2}^{j_1,j_2} + \sqrt{(j_2-m_2)(j_2+m_2+1)}e_{m_1,m_2+1}^{j_1,j_2}\\
&\Delta(Z_-)e_{m_1,m_2}^{j_1,j_2} = \sqrt{(j_1+m_1)(j_1-m_1+1)}e_{m_1-1,m_2}^{j_1,j_2} + \sqrt{(j_2+m_2)(j_2-m_2+1)}e_{m_1,m_2-1}^{j_1,j_2}.
\end{aligned} \tag{5.4}$$

Thus the action of $\Delta(H)$, $\Delta(Z_\pm)$ on $e_{m_1,m_2}^{j_1,j_2}$ is the same as the action of $\delta(h)$, $\delta(z_\pm)$ on $e_{m_1}^{j_1} \otimes e_{m_2}^{j_2}$. Consequently, we can write

$$e_m^{(j_1j_2)j} = \sum_{m_1+m_2=m} C_{m_1,m_2,m}^{j_1,j_2,j} e_{m_1,m_2}^{j_1,j_2} \tag{5.5}$$

with $C_{m_1,m_2,m}^{j_1,j_2,j}$ the classical $su(2)$ Clebsch–Gordan coefficients given by [12]

$$\begin{aligned}
C_{m_1,m_2,m}^{j_1,j_2,j} = \delta_{m_1+m_2,m}&\left(\frac{(j_1+j_2-j)!(j_1-j_2+j)!(-j_1+j_2+j)!}{(j_1+j_2+j+1)!}\right)^{1/2}\\
&\times((j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j+m)!(j-m)!(2j+1))^{1/2}\\
&\times\sum_k(-1)^k/(k!(j_1+j_2-j-k)!(j_1-m_1-k)!(j_2+m_2-k)!\\
&\times(j-j_2+m_1+k)!(j-j_1-m_2+k)!).
\end{aligned} \tag{5.6}$$

The action of $\Delta(H)$ and $\Delta(Z_\pm)$ is then indeed

$$\begin{aligned}
&\Delta(H)e_m^{(j_1j_2)j} = 2me_m^{(j_1j_2)j}\\
&\Delta(Z_\pm)e_m^{(j_1j_2)j} = \sqrt{(j\mp m)(j\pm m+1)}\,e_{m\pm1}^{(j_1j_2)j}
\end{aligned} \tag{5.7}$$

i.e. the same as the standard action (2.5) of H and $Z_\pm$ on a basis e_m^j. Consequently, also for $\Delta(X)$ and $\Delta(Y)$ the action on $e_m^{(j_1j_2)j}$ is the same as the standard action of X and Y on a basis e_m^j, and the $e_m^{(j_1j_2)j}$ are genuinely 'coupled states' for $\mathcal{U}_h(sl(2))$. The decomposition of the tensor product for $\mathcal{U}_h(sl(2))$ is the same as in $su(2)$:

$$V^{(j_1)}\otimes V^{(j_2)} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} V^{(j)}. \tag{5.8}$$

From (5.2) and (5.5), it follows that we can write

$$\begin{aligned}
e_m^{(j_1j_2)j} &= \sum_{m_1+m_2=m} C_{m_1,m_2,m}^{j_1,j_2,j}\sum_{k,l\geqslant 0} A_{k,l}^{m_1,m_2} e_{m_1+k}^{j_1}\otimes e_{m_2+l}^{j_2}\\
&= \sum_{n_1,n_2}\left(\sum_{m_1+m_2=m} C_{m_1,m_2,m}^{j_1,j_2,j} A_{n_1-m_1,n_2-m_2}^{m_1,m_2}\right) e_{n_1}^{j_1}\otimes e_{n_2}^{j_2}.
\end{aligned} \tag{5.9}$$

Thus we have the following.

Theorem 7. The Clebsch–Gordan coefficients for $\mathcal{U}_h(sl(2))$, in

$$e_m^{(j_1j_2)j} = \sum_{n_1,n_2} \mathcal{C}_{n_1,n_2,m}^{j_1,j_2,j}(h)\, e_{n_1}^{j_1}\otimes e_{n_2}^{j_2} \tag{5.10}$$

are given by

$$\mathcal{C}_{n_1,n_2,m}^{j_1,j_2,j}(h) = \sum_{m_1+m_2=m} C_{m_1,m_2,m}^{j_1,j_2,j} A_{n_1-m_1,n_2-m_2}^{m_1,m_2} \tag{5.11}$$

with $C_{m_1,m_2,m}^{j_1,j_2,j}$ the usual $sl(2)$ Clebsch–Gordan coefficients, and $A_{n_1-m_1,n_2-m_2}^{m_1,m_2}$ determined by (5.3) and (4.7).

One question that naturally arises is whether these Clebsch–Gordan coefficients satisfy an orthogonality relation. The answer is negative. In order to have an orthogonality relation, one needs a $*$-Hopf algebra [13, section 4.1.F]. For $\mathcal{U}_h(sl(2))$ the obvious choice of the $*$-operation would be that induced from $H^* = H$, $Z_\pm^* = Z_\mp$ (making the e-basis of $V^{(j)}$ an orthonormal basis). However, this $*$-operation is not compatible with the coalgebra structure.

Nevertheless, we have recently showed [14] that the present Clebsch–Gordan coefficients do satisfy orthogonality-like relations, namely

$$\begin{aligned}
&\sum_{n_1,n_2}(-1)^{j_1+j_2-j}\mathcal{C}_{n_1,n_2,m}^{j_1,j_2,j}(h)\mathcal{C}_{-n_1,-n_2,-m'}^{j_1,j_2,j'}(h) = \delta_{j,j'}\delta_{m,m'}\\
&\sum_{j,m}(-1)^{j_1+j_2-j}\mathcal{C}_{n_1,n_2,m}^{j_1,j_2,j}(h)\mathcal{C}_{-n_1',-n_2',-m}^{j_1,j_2,j}(h) = \delta_{n_1,n_1'}\delta_{n_2,n_2'}.
\end{aligned}$$

These relations were derived using special properties of the auxiliary coefficients $A_{k,l}^{m_1,m_2}$ and the ordinary orthogonality relations for $su(2)$ Clebsch–Gordan coefficients; for more details see [14].

We end this section by noting that $\mathcal{U}_h(sl(2))$, and its Clebsch–Gordan coefficients, also have another interpretation. Due to the invertible nonlinear map described in section 2, one can identify the algebra part of $\mathcal{U}_h(sl(2))$ with $U(sl(2))$. In other words, the Hopf algebra one is dealing with has $U(sl(2))$ as algebra structure, with generators H, $Z_\pm$, and defining relations (2.4). There is no deformation in the algebra part. The deformation comes in at the level of the coalgebra part: see, for example, (4.3) and (4.5). So, roughly speaking we are dealing with the ordinary $sl(2)$ algebra and its usual finite-dimensional irreducible representations, but equipped with a deformed comultiplication. The deformation of the Clebsch–Gordan coefficients then stems from this deformed coproduct.

6. Examples and conclusion

The formula (5.11) allows one to calculate the $\mathcal{U}_h(sl(2))$ Clebsch–Gordan coefficients for arbitrary parameters, since the usual $sl(2)$ coefficients $C^{j_1,j_2,j}_{m_1,m_2,m}$ are known, and the coefficients $A^{m_1,m_2}_{n_1-m_1,n_2-m_2}$ are determined in this paper. For example, one finds

$$\mathcal{C}^{2,2,3}_{2,0,2}(h) = C^{2,2,3}_{2,0,2} = 1/\sqrt{2}$$
$$\mathcal{C}^{2,2,3}_{2,0,3}(h) = 0$$
$$\mathcal{C}^{2,2,3}_{2,0,-1}(h) = -6h^3 C^{2,2,3}_{0,-1,-1} - 4\sqrt{6}h^3 C^{2,2,3}_{1,-2,-1} = -18h^3/\sqrt{5}.$$

More generally, suppose $m = n_1 + n_2 + p$, then (5.11) becomes

$$\mathcal{C}^{j_1,j_2,j}_{n_1,n_2,m}(h) = \sum_{m_1} C^{j_1,j_2,j}_{m_1,m-m_1,m} A^{m_1,m-m_1}_{n_1-m_1,-n_1+m_1-p}. \tag{6.1}$$

First, let $p = 0$. Since $A^{m_1,m_2}_{k,l}$ is zero if k or l are negative, it follows that the only contribution in (6.1) is for $m_1 = n_1$, and with $A^{m_1,m_2}_{0,0} = 1$ it follows that $\mathcal{C}^{j_1,j_2,j}_{n_1,n_2,n_1+n_2}(h) = C^{j_1,j_2,j}_{n_1,n_2,n_1+n_2}$. Next, suppose that $p > 0$, then one can see that at least one of the indices of A in (6.1) is negative, thus in this case $\mathcal{C}^{j_1,j_2,j}_{n_1,n_2,n_1+n_2+p}(h) = 0$. Finally, suppose that $p < 0$. Now there can be a number of contributions in (6.1), and by (5.3) and (4.7) they all have the same power of h, namely h^{-p}. Thus we have the following property.

Proposition 8. The Clebsch–Gordan coefficients for $\mathcal{U}_h(sl(2))$ satisfy

- if $m = n_1 + n_2$ then $\mathcal{C}^{j_1,j_2,j}_{n_1,n_2,m}(h) = C^{j_1,j_2,j}_{n_1,n_2,m}$;
- if $m > n_1 + n_2$ then $\mathcal{C}^{j_1,j_2,j}_{n_1,n_2,m}(h) = 0$;
- if $m < n_1 + n_2$ then $\mathcal{C}^{j_1,j_2,j}_{n_1,n_2,m}(h)$ is a monomial in $h^{n_1+n_2-m}$.

This proposition implies that the Clebsch–Gordan coefficients $\mathcal{C}^{j_1,j_2,j}_{n_1,n_2,m}(h)$ are simple deformations of the $sl(2)$ Clebsch–Gordan coefficients in the sense that for $h = 0$ one has $\mathcal{C}^{j_1,j_2,j}_{n_1,n_2,m}(0) = C^{j_1,j_2,j}_{n_1,n_2,m}$. As is well known, the $sl(2)$ or $su(2)$ representations play a fundamental role in many physical models and theories, and so do their Clebsch–Gordan coefficients. It would be interesting to investigate whether the present Clebsch–Gordan coefficients of $\mathcal{U}_h(sl(2))$ still have a physical interpretation in corresponding deformed models or theories. The explicit formula given here, together with the properties mentioned, should prove to be very helpful in such an investigation.

Appendix

Here, we give proofs of the combinatorial identities in lemmas 1, 2 and 4. The terms appearing in the sums of these lemmas are hypergeometric terms [11]. The sums in lemmas 1 and 2 are definite sums, i.e. the summation limit also appears in the summand. For such sums, Zeilberger's algorithm [11, ch 6], which is also known as the method of creative telescoping [15], can be used to find a recurrence relation. A Mathematica implementation `Zb` of Zeilberger's algorithm can, for example, be found in the package `Zb.m` [16]. Considering the sum of lemma 1,

$$f(k) = \sum_{n=1}^{k} (-1)^n \frac{(1/2)_n}{n!} \binom{k-1}{n-1} \tag{A.1}$$

Zeilberger's algorithm yields the following recurrence relation for $f(k)$:

$$2(k+1) f(k+1) = (2k-1) f(k) \tag{A.2}$$

with the initial condition $f(1) = -1/2$. The closed form expression for $f(k)$ then easily follows and is given in lemma 1.

Lemma 2 is similar, with

$$f_n(s) = \sum_{k=1}^{s-1} \frac{(2k-2)!}{k!(k-1)!} \frac{(2s-2k-2)!}{(s-k)!(s-k-1)!} k^n \tag{A.3}$$

where $n = 0$, 1 or 2. The recurrence relations obtained by Zeilberger's algorithm (or by the program `Zb`) read

$$4(s-1) f_0(s) - (s+1) f_0(s+1) + \frac{2(2s-2)!}{s!(s-1)!} = 0 \tag{A.4}$$

$$4(s-1) f_1(s) - s f_1(s+1) + \frac{(2s-2)!}{(s-1)!^2} = 0 \tag{A.5}$$

$$4 f_2(s) - f_2(s+1) + \frac{(2s-2)!}{s!(s-2)!} = 0. \tag{A.6}$$

With the initial conditions $f_n(2) = 1$, the closed form expressions for $f_n(s)$ given in lemma 2 are deduced from these recurrence relations.

The statement in lemma 4 is rather different, in the sense that this is an indefinite summation (i.e. the upper limit does not appear in $g(l)$). The term $g(l)$ is again a hypergeometric term however. For such summations Gosper's algorithm [11, 17] decides whether the sum can be written in closed form, and also gives the closed form if it exists. The summation formula in lemma 4 is a direct output of the Mathematica program `Gosper` of the package `Zb.m`.

References

[1] Manin Yu I 1988 *Quantum Groups and Non-commutative Geometry* (Montreal: Les Publications du CRM)
[2] Schupp P, Watts P and Zumino B 1992 *Lett. Math. Phys.* **25** 139
[3] Dubois-Violette M and Launer G 1990 *Phys. Lett.* **245B** 175
[4] Kupershmidt B A 1992 *J. Phys. A: Math. Gen.* **25** L1239
[5] Zakrzewski S 1991 *Lett. Math. Phys.* **22** 287
[6] Ohn Ch 1992 *Lett. Math. Phys.* **25** 85
[7] Ballesteros A and Herranz F J 1996 *J. Phys. A: Math. Gen.* **29** L311
[8] Dobrev V K 1996 Representations of the Jordanian quantum algebra $\mathcal{U}_h(sl(2))$ *Preprint* IC/96/14, ICTP, Trieste
[9] Abdesselam B, Chakrabarti A and Chakrabarti R 1996 *Mod. Phys. Lett.* A **11** 2883
[10] Aizawa N 1997 *J. Phys. A: Math. Gen.* **30** 5981
[11] Petkovšek M, Wilf H S and Zeilberger D 1996 *A=B* (Wellesley, MA: Peters)
[12] Edmonds A R 1960 *Angular Momentum in Quantum Mechanics* (Princeton: Princeton University Press)
[13] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)

[14] Van der Jeugt J 1997 The Jordanian deformation of su(2) and Clebsch–Gordan coefficients *Preprint* q-alg/9709005
[15] Zeilberger D 1991 *J. Symb. Comput.* **11** 195
[16] Paule P and Schorn M 1995 *J. Symb. Comput.* **20** 673. Programs obtainable from `http://info.risc.uni-linz.ac.at:70/labs-info/comblab/software/Summation/PauleSchorn.index.html`.
[17] Gosper R W Jr 1978 *Proc. Natl Acad. Sci., USA* **75** 40